Examples of signature $(2, 2)$ manifolds with commuting curvature operators

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Abstract
We exhibit Walker manifolds of signature $(2, 2)$ with various commutativity properties for the Ricci operator, the skew-symmetric curvature operator and the Jacobi operator. If the Walker metric is a Riemannian extension of an underlying affine structure $\mathcal{A}$, these properties are related to the Ricci tensor of $\mathcal{A}$.

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1. Introduction

Let $\mathcal{M} := (M, g)$ be a pseudo-Riemannian manifold of signature (p, q), ∇ be the associated Levi-Civita connection and $\mathcal{R}(x, y) := \nabla_x \nabla_y - \nabla_y \nabla_x - \nabla_{[x,y]}$ be the curvature operator. With our sign convention, the *Jacobi operator* is given by $\mathcal{J}(x) : y \to \mathcal{R}(y, x)x$. Let ρ be the associated Ricci operator; $g(\rho x, x) = \text{Tr}\{\mathcal{J}(x)\}$. We shall study relations between the algebraic properties of the curvature operator and the underlying geometry of the manifold. Commutativity conditions of curvature operators have been considered extensively in the study of submanifolds (see, for example, [31, 35]), hence it is natural to look at them from a broader intrinsic point of view.

Definition 1.1. $\mathcal{M}$ *is said to be*

(1) Einstein if ρ is a scalar multiple of the identity,
(2) Jacobi–Ricci commuting if $\mathcal{J}(x)\rho = \rho\mathcal{J}(x) \,\forall x$,
(3) curvature-Ricci commuting if $\mathcal{R}(x, y)\rho = \rho\mathcal{R}(x, y) \,\forall x, y$,
(4) Jacobi–Jacobi commuting if $\mathcal{J}(x)\mathcal{J}(y) = \mathcal{J}(y)\mathcal{J}(x) \,\forall x, y$,
(5) curvature-Jacobi commuting if $\mathcal{J}(x)\mathcal{R}(y, z) = \mathcal{R}(y, z)\mathcal{J}(x) \,\forall x, y, z$,
(6) curvature–curvature commuting if $\mathcal{R}(w, x)\mathcal{R}(y, z) = \mathcal{R}(y, z)\mathcal{R}(w, x) \,\forall w, x, y, z$.

Commutativity properties of the skew-symmetric curvature operator and of the Jacobi operator were first studied in the Riemannian setting by Tsankov [35]. He showed that if $\mathcal{M}$ is a hypersurface in $\mathbb{R}^{m+1}$ with $\mathcal{J}(x)\mathcal{J}(y) = \mathcal{J}(y)\mathcal{J}(x)$ for all $x \perp y$, then necessarily $\mathcal{M}$ had constant sectional curvature; this result was subsequently extended to the general Riemannian context in [7] and additional results were obtained in the general pseudo-Riemannian setting in [8, 10, 24]. Tsankov also derived results relating to hypersurfaces where $\mathcal{R}(w, x)\mathcal{R}(y, z) = \mathcal{R}(y, z)\mathcal{R}(w, x)$. Videv studied manifolds where $\rho \mathcal{J}(x) = \mathcal{J}(x)\rho$ for all x. Many of these investigations were originally suggested by Stanilov [33, 34].

The conditions in definition 1.1 have also been described elsewhere in the literature as 'Jacobi–Videv', 'skew-Videv', 'Jacobi–Tsankov', 'mixed-Tsankov' and 'skew-Tsankov', respectively, and the general field of investigation of such conditions is often referred to as Stanilov–Tsankov–Videv theory. The curvature-Ricci commuting condition has also been denoted as 'Ricci semi-symmetric'; it is a generalization of the semi-symmetric condition (see [1] and the references therein). Semi-symmetric manifolds of conullity two are curvature–curvature commuting [9, 11]. We have chosen to change the notation from that employed previously to put these conditions in parallel as much as possible.

In this paper, we shall exhibit families of manifolds having some, but not necessarily all, of these properties in order to obtain insight into relationships between these concepts and the underlying geometry. We shall work with a restricted class of Walker manifolds of signature $(2, 2)$; this class is both sufficiently rich to offer a significant number of examples and sufficiently restricted to permit a relatively complete analysis. We have not attempted to obtain the most general possible classification results for Walker signature $(2, 2)$ manifolds as our experience in similar related problems is that these tend to be excessively technical; for example, there is as yet no classification of Einstein Walker signature $(2, 2)$ manifolds and there is as yet no classification of anti-self-dual Walker signature $(2, 2)$ manifolds. As the family we shall examine has been studied extensively in other contexts [5, 6], we can also relate curvature commutativity properties for these manifolds to other properties such as Einstein, self-dual, anti-self-dual and Osserman.

Walker [36] studied pseudo-Riemannian manifolds with a parallel field of null planes and derived a canonical form. Lorentzian Walker metrics have been studied extensively in the physics literature since they constitute the background metric of the pp-wave models ([2, 25, 26, 30] to list a few of the many possible references; the literature is a vast one); a pp-wave spacetime admits a covariantly constant null vector field U and therefore it is trivially recurrent (i.e., $\nabla U = \omega \otimes U$ for some one-form ω). Lorentzian Walker metrics present many specific features both from the physical and geometric viewpoints [12, 14, 27, 32]. We also refer to the related work of Hall [21] and Hall and da Costa [22] for generalized Lorentzian Walker manifolds (i.e. for spacetimes admitting a nonzero vector field n^a satisfying $R_{ijkl}n^l = 0$ or admitting a rank 2-symmetric or anti-symmetric tensor H_{ab} with $\nabla H = 0$).

One says that a pseudo-Riemannian manifold $\mathcal{M}$ of signature $(2, 2)$ is a *Walker manifold* if it admits a parallel totally isotropic 2-plane field; see [13, 29] for further details. Such a manifold is locally isometric to an example of the following form: $\mathcal{M} := (\mathcal{O}, g)$ where $\mathcal{O}$ is an open subset of $\mathbb{R}^4$ and where the metric is given by

$$g(\partial_{x_1}, \partial_{x_3}) = g(\partial_{x_2}, \partial_{x_4}) = 1,$$
$$g(\partial_{x_i}, \partial_{x_j}) = g_{ij}(x_1, x_2, x_3, x_4), \qquad \text{for} \quad i, j = 3, 4; \tag{1a}$$

here (x_1, x_2, x_3, x_4) are the coordinates on $\mathbb{R}^4$. In this paper, we shall examine the concepts of definition 1.1 for a restricted category of signature $(2, 2)$ Walker metrics where we set $g_{33} = g_{44} = 0$:

$$g(\partial_{x_1}, \partial_{x_3}) = g(\partial_{x_2}, \partial_{x_4}) = 1, \qquad g(\partial_{x_3}, \partial_{x_4}) = g_{34}(x_1, x_2, x_3, x_4). \tag{1b}$$

Let $dx_1 dx_2 dx_3 dx_4$ orient $\mathbb{R}^4$. The study of self-dual and anti-self-dual metrics is crucial in Lorentzian geometry, see, for example, [3, 23]. The same is true in the higher signature context [4, 15, 17, 23, 28]. One says that $\mathcal{M}$ is *Osserman* if the spectrum of the Jacobi operator is constant on the pseudo-sphere of unit spacelike vectors or, equivalently, on the pseudo-sphere of unit timelike vectors. The notion of *conformally Osserman* is defined using the conformal Jacobi operator. One has that $\mathcal{M}$ is conformally Osserman $\Leftrightarrow$ $\mathcal{M}$ is either self-dual or anti-self-dual [5]. If $f = f(x_1, x_2, x_3, x_4)$, let $f_{/i} := \partial_{x_i} f$ and $f_{/ij} := \partial_{x_i} \partial_{x_j} f$. One has the following surprising result:

Theorem 1.2. *Let the metric be as in equation (1b). Then*

(1) $\mathcal{M}$ is self-dual $\Leftrightarrow$ $g_{34} = x_1 p(x_3, x_4) + x_2 q(x_3, x_4) + s(x_3, x_4)$.
(2) $\mathcal{M}$ is anti-self-dual $\Leftrightarrow$ $g_{34} = x_1 p(x_3, x_4) + x_2 q(x_3, x_4) + s(x_3, x_4) + \xi(x_1, x_4) + \eta(x_2, x_3)$ with $p_{/3} = q_{/4}$ and $g_{34} p_{/3} - x_1 p_{/34} - x_2 p_{/33} - s_{/34} = 0$.
(3) The following assertions are equivalent:

   (a) $\mathcal{M}$ is Osserman.
   (b) $\mathcal{M}$ is Einstein.
   (c) $g_{34} = x_1 p(x_3, x_4) + x_2 q(x_3, x_4) + s(x_3, x_4)$, where $p^2 = 2p_{/4}, q^2 = 2q_{/3}$ and $pq = p_{/3} + q_{/4}$.
   (d) $\rho = 0$.

We emphasize that it is a crucial feature of these examples that the Ricci flat, Einstein and Osserman are equivalent conditions; this is not the case, of course, for general Walker metrics of signature $(2, 2)$.

The conditions on p and q which are given in assertion $(3c)$ of theorem 1.2 will play an important role in what follows. The following is a useful technical result that will be central in our discussions:

Lemma 1.3. *Let $\mathcal{O}$ be an open connected subset of $\mathbb{R}^4$. Let $p, q \in C^\infty(\mathcal{O})$ be functions only of (x_3, x_4). Then the following conditions are equivalent:*

(1) $p^2 = 2p_{/4}, q^2 = 2q_{/3}$ and $pq = p_{/3} + q_{/4}$.
(2) $p^2 = 2p_{/4}, q^2 = 2q_{/3}$ and $p_{/3} = q_{/4} = \frac{1}{2} pq$.
(3) There exist $(a_0, a_3, a_4) \in \mathbb{R}^3 - \{0\}$ so that $p = -2a_4(a_0 + a_3 x_3 + a_4 x_4)^{-1}$ and $q = -2a_3(a_0 + a_3 x_3 + a_4 x_4)^{-1}$.

The Jacobi–Ricci commuting and curvature-Ricci commuting are equivalent concepts in the context of metrics given by equation (1b).

Theorem 1.4. *Let $\mathcal{M}$ be given by equation (1b). The following assertions are equivalent:*

(1) $\mathcal{M}$ is Jacobi–Ricci commuting.
(2) $\mathcal{M}$ is curvature-Ricci commuting.
(3) $g_{34} = x_1 p(x_3, x_4) + x_2 q(x_3, x_4) + s(x_3, x_4)$ where $p_{/3} = q_{/4}$.

Lemma 1.3 shows that the conditions of theorem 1.2 $(3c)$ are very rigid. On the other hand, the condition of theorem 1.4 (3) that $p_{/3} = q_{/4}$ is, of course, nothing but the condition that $\omega := p\, dx_4 + q\, dx_3$ is a closed one-form and thus there are many examples.

We now turn our attention to Tsankov theory. A Riemannian or Lorentzian manifold is Jacobi–Jacobi commuting if and only if it is of constant sectional curvature [7, 8]. This is not the case in the higher signature context. Further observe that any Jacobi–Jacobi commuting metric given by equation (1b) is semi-symmetric since the Jacobi operators are two-step

nilpotent [19]. Also observe that curvature-Ricci and curvature–curvature commuting are equivalent conditions for metrics (1*b*), which is not a general fact (see theorem 1.7 and remark 1.9).

Theorem 1.5. *Let $\mathcal{M}$ be given by equation (1b).*

(1) The following assertions are equivalent:
  (a) $\mathcal{M}$ is curvature–curvature commuting.
  (b) $g_{34} = x_1 p(x_3, x_4) + x_2 q(x_3, x_4) + s(x_3, x_4)$ where $p_{/3} = q_{/4}$.

(2) Let $\mathcal{P} := \mathrm{Span}\{\partial_{x_1}, \partial_{x_2}\}$. The following assertions are equivalent:
  (a) $\mathcal{R}(x, y)z \in \mathcal{P}$ for all x, y, z and $\mathcal{R}(x, y)z = 0$ if x, y or z is in $\mathcal{P}$.
  (b) $\mathcal{M}$ is curvature-Jacobi commuting.
  (c) $\mathcal{M}$ is Jacobi–Jacobi commuting.
  (d) $\mathcal{J}(x)^2 = 0$ for all x.
  (e) $\rho = 0$.
  (f) $g_{34} = x_1 p(x_3, x_4) + x_2 q(x_3, x_4) + s(x_3, x_4)$ where $p^2 = 2p_{/4}, q^2 = 2q_{/3}$ and $pq = p_{/3} + q_{/4}$.

As it is a feature of our analysis that the warping function g_{34} is necessarily affine, it is worth putting such metrics in a geometrical context. Let ∇ be a torsion-free connection on a smooth manifold N; the resulting structure $\mathcal{A} := (N, \nabla)$ is said to be an *affine manifold*. The associated *Jacobi operator* and *Ricci tensor* are defined, respectively, by

$$\mathcal{J}_{\mathcal{A}}(x) : y \to \mathcal{R}_{\mathcal{A}}(y, x)x \qquad \text{and} \qquad \rho_{\mathcal{A}}(x, y) := \mathrm{Tr}\{z \to \mathcal{R}_{\mathcal{A}}(z, x)y\}.$$

We say $\mathcal{A}$ is *affine Osserman* if $\mathcal{J}_{\mathcal{A}}(x)$ is nilpotent for all tangent vectors x, i.e. if $\mathrm{Spec}\{\mathcal{J}_{\mathcal{A}}(x)\} = \{0\}$ for all tangent vectors. If the affine structure arises from a pseudo-Riemannian structure, i.e. if ∇ is the Levi-Civita connection of a pseudo-Riemannian metric, then (N, ∇) is affine Osserman implies (N, g) is Osserman; the converse implication is false in general as not every Osserman manifold is nilpotent Osserman. If $x = (x_1, \ldots, x_m)$ are local coordinates on N, the Christoffel symbols are given by $\nabla_{\partial_{x_i}} \partial_{x_j} = \sum_k \Gamma_{ij}{}^k \partial_{x_k}$.

Let $\mathcal{A} := (N, \nabla)$ be a two-dimensional affine manifold. Let (x_3, x_4) be local coordinates on N. Let $\omega = x_1 dx_3 + x_2 dx_4 \in T^*N$; (x_1, x_2) are the dual fiber coordinates. Let $\xi = \xi_{ij}(x_3, x_4) \in C^\infty(S^2(T^*N))$ be an auxiliary symmetric bilinear form. The *deformed Riemannian extension* is the Walker metric on T^*N defined by setting

$$
\begin{aligned}
& g(\partial_{x_1}, \partial_{x_3}) = g(\partial_{x_2}, \partial_{x_4}) = 1, \\
& g(\partial_{x_3}, \partial_{x_3}) = -2x_1 \Gamma_{33}{}^3(x_3, x_4) - 2x_2 \Gamma_{33}{}^4(x_3, x_4) + \xi_{33}(x_3, x_4), \\
& g(\partial_{x_3}, \partial_{x_4}) = -2x_1 \Gamma_{34}{}^3(x_3, x_4) - 2x_2 \Gamma_{34}{}^4(x_3, x_4) + \xi_{34}(x_3, x_4), \\
& g(\partial_{x_4}, \partial_{x_4}) = -2x_1 \Gamma_{44}{}^3(x_3, x_4) - 2x_2 \Gamma_{44}{}^4(x_3, x_4) + \xi_{44}(x_4, x_4).
\end{aligned}
\tag{1c}
$$

The crucial fact [37] is that the resulting neutral signature pseudo-Riemannian manifold $\mathcal{M}$ is independent of the particular coordinates (x_3, x_4) which were chosen and is determined by (N, ∇, ξ). Moreover, proceeding as in [18] one has that (N, ∇) is affine Osserman if and only if the deformed Riemannian extension is Osserman for any choice of ξ.

Assuming that $g_{33} = g_{44} = 0$ on M is equivalent to assuming that

$$\Gamma_{33}{}^3 = \Gamma_{44}{}^3 = \Gamma_{33}{}^4 = \Gamma_{44}{}^4 = \xi_{33} = \xi_{44} = 0 \tag{1d}$$

on N, i.e. there exist coordinates on N where the two families of coordinate lines on N are parallel and ξ-null.

We use the correspondence between $\mathcal{A}$ and $\mathcal{M}$ to express the conditions which appear in theorems 1.2, 1.4 and 1.5 in a natural and covariant setting.

Theorem 1.6. *Let $\mathcal{A}$ be a two-dimensional affine manifold satisfying equation (1d). Let $\mathcal{M}$ be the deformed Riemannian extension defined by equation (1c). Decompose $\rho_{\mathcal{A}} = \rho_{\mathcal{A}}^s + \rho_{\mathcal{A}}^a$ into the symmetric and anti-symmetric parts. Then*

(1) $\rho_{\mathcal{A}}^a = 0 \Leftrightarrow \mathcal{M}$ is curvature–curvature commuting $\Leftrightarrow \mathcal{M}$ is curvature-Ricci commuting $\Leftrightarrow \mathcal{M}$ is Jacobi–Ricci commuting.
(2) $\rho_{\mathcal{A}}^s = 0 \Leftrightarrow \rho_{\mathcal{A}} = 0 \Leftrightarrow \mathcal{A}$ is affine Osserman $\Leftrightarrow \mathcal{M}$ is Osserman $\Leftrightarrow \mathcal{M}$ is curvature-Jacobi commuting $\Leftrightarrow \mathcal{M}$ is Jacobi–Jacobi commuting.

If $g_{33} = g_{44} = 0$, then $\rho_{\mathcal{A}}^s = 0 \Leftrightarrow \rho_{\mathcal{A}} = 0$. This is, of course, a reflection of the equivalence of conditions (1) and (2) in lemma 1.3; so far we have only been considering two different conditions on g_{34}. However, this is not the case for a more general affine extension. The following is the analogue of theorem 1.6 in the more general context; in contrast to the situation with theorem 1.6, there are four cases of interest and not just two. The following result extends theorem 1.6 to the general covariant setting of the cotangent bundle of a two-dimensional manifold:

Theorem 1.7. *Let $\mathcal{A}$ be a two-dimensional affine manifold and $\mathcal{M}$ be the deformed Riemannian extension defined by equation (1c); we impose no additional restrictions on ∇. Then*

(1) $\rho_{\mathcal{A}}^a = 0 \Leftrightarrow \mathcal{M}$ is curvature–curvature commuting.
(2) $\rho_{\mathcal{A}}^s = 0 \Leftrightarrow \mathcal{A}$ is affine Osserman $\Leftrightarrow \mathcal{M}$ is Osserman.
(3) $\rho_{\mathcal{A}}^a = 0$ or $\rho_{\mathcal{A}}^s = 0 \Leftrightarrow \mathcal{M}$ is curvature-Ricci commuting $\Leftrightarrow \mathcal{M}$ is Jacobi–Ricci commuting.
(4) $\rho_{\mathcal{A}} = 0 \Leftrightarrow \mathcal{M}$ is curvature-Jacobi commuting $\Leftrightarrow \mathcal{M}$ is Jacobi–Jacobi commuting.

Remark 1.8. If ∇ is the torsion-free connection on $\mathbb{R}^2$ with nonzero Christoffel symbols $\nabla_{\partial_{x_3}}\partial_{x_4} = \nabla_{\partial_{x_4}}\partial_{x_3} = f(x_3)\partial_{x_3}, \nabla_{\partial_{x_4}}\partial_{x_4} = f(x_3)\partial_{x_4}$, for $f = f(x_3)$ with $\dot{f}(x_3) \neq 0$, we have $\rho_{\mathcal{A}}^s = 0$ while $\rho_{\mathcal{A}}^a \neq 0$. Moreover, for the choice $\nabla_{\partial_{x_3}}\partial_{x_3} = f(x_3, x_4)\partial_{x_4}$, for $f = f(x_3, x_4)$ with $f_{/4} \neq 0$, it follows that $\rho_{\mathcal{A}}^s \neq 0$ while $\rho_{\mathcal{A}}^a = 0$. Thus, in contrast to the case studied in theorem 1.6, these conditions are distinct. Combined with our previous results, this shows the four possibilities in theorem 1.7 are distinct.

Remark 1.9. Adopt the notation of equation (1b), one now has that any of the conditions Osserman, Einstein, curvature–curvature commuting, curvature-Jacobi commuting, Jacobi–Jacobi commuting, curvature-Ricci commuting or Jacobi–Ricci commuting implies that g_{34} is affine in the variables $\{x_1, x_2\}$ and hence $\mathcal{M}$ is a Riemannian extension. This is not the case, however, in the more general context of equation (1a). Indeed, let $\mathcal{M}$ have the form given in equation (1a) where

(1) $g_{33} = 4kx_1^2 - \frac{1}{4k}f(x_4)^2$, $g_{44} = 4kx_2^2$ and $g_{34} = 4kx_1x_2 + x_2 f(x_4) - \frac{1}{4k}\dot{f}(x_4)$ for $f = f(x_4)$ non-constant and for $k \neq 0$. Then, $\mathcal{M}$ is Osserman with eigenvalues $\{0, 4k, k, k\}$ and $\rho \neq 0$. The Jacobi operators are diagonalizable at $P \Leftrightarrow 24kf(x_4)\dot{f}(x_4)x_2 - 12k\ddot{f}(x_4)x_1 + 3f(x_4)\ddot{f}(x_4) + 4\dot{f}(x_4)^2 = 0$, $\mathcal{M}$ is Jacobi–Ricci commuting and curvature-Ricci commuting, $\mathcal{M}$ is neither Jacobi–Jacobi commuting nor curvature-Jacobi commuting nor curvature–curvature commuting.

(2) $g_{33} = x_1x_2$, $g_{44} = -x_1x_2$ and $g_{34} = (x_2^2 - x_1^2)/2$. Then, $\mathcal{M}$ is curvature–curvature commuting, curvature-Ricci commuting, Jacobi–Ricci commuting and $\rho^2 = -\mathrm{id}$. However, $\mathcal{M}$ is neither Einstein nor curvature-Jacobi commuting nor Jacobi–Jacobi commuting.

(We refer to [16] for the proof of assertion (1) and to [20] for the proof of assertion (2).)

Here is a brief outline to the paper. In section 2, we reduce the proof of theorems 1.2, 1.4 and 1.5 to the case where g_{34} is affine in $\{x_1, x_2\}$. In section 3, we study the Osserman

condition to establish theorem 1.2. We then turn to the study of commutativity conditions. In section 4, we establish theorem 1.4 and in section 5, we verify theorem 1.5. Section 6 deals with affine extensions and the proof of theorem 1.6. Finally, in the appendix, we prove the technical result stated in lemma 1.3. We shall omit the proof of theorem 1.7 as the proof is similar to the proof we shall give to establish theorem 1.6; details are available from the authors upon request.

2. Reduction to an affine warping function

Let $\mathcal{M}$ be given by equation (1*b*). One of the crucial features we shall exploit is that ρ, $\mathcal{J}(x)$ and $\mathcal{R}(x)$ are polynomials in the jets of g_{34}. One has by [5] that the nonzero components of the curvature tensor are, after adjusting for a difference in the sign convention used therein, given by

$$R_{1334} = -\tfrac{1}{4}(g_{34/1}g_{34/2} - 2g_{34/13}), \qquad R_{1314} = \tfrac{1}{2}g_{34/11},$$

$$R_{1434} = -\tfrac{1}{4}\left(-g_{34/1}^2 + 2g_{34/14}\right), \qquad R_{1324} = \tfrac{1}{2}g_{34/12},$$

$$R_{2334} = -\tfrac{1}{4}\left(g_{34/2}^2 - 2g_{34/23}\right), \qquad R_{1423} = \tfrac{1}{2}g_{34/12},$$

$$R_{2434} = -\tfrac{1}{4}(-g_{34/1}g_{34/2} + 2g_{34/24}), \qquad R_{2324} = \tfrac{1}{2}g_{34/22},$$

$$R_{3434} = -\tfrac{1}{2}(-g_{34}g_{34/1}g_{34/2} + 2g_{34/34}).$$

One can now use the metric to raise indices and compute $\mathcal{J}$, $\mathcal{R}$ and ρ.

If P is a polynomial and U is a monomial expression, we let $c(P, U)$ be the coefficient of U in P. Let

$$\mathcal{R}_1 := \mathcal{R}\left(\sum_i a_i\partial_{x_i}, \sum_j b_j\partial_{x_j}\right), \qquad \mathcal{R}_2 := \mathcal{R}\left(\sum_i c_i\partial_{x_i}, \sum_j d_j\partial_{x_j}\right),$$

$$\mathcal{J}_1 := \mathcal{J}\left(\sum_i v_i\partial_{x_i}\right), \qquad \mathcal{J}_2 := \mathcal{J}\left(\sum_i w_i\partial_{x_i}\right). \tag{2a}$$

We used Mathematica to assist us in the following computations. One has

$$\rho_{21} = \tfrac{1}{2}g_{34/11}, \qquad \rho_{12} = \tfrac{1}{2}g_{34/22},$$

$$c(\{\rho\mathcal{R}_1 - \mathcal{R}_1\rho\}_{21}, a_4b_1) = -\tfrac{1}{4}g_{34/11}^2,$$

$$c(\{\rho\mathcal{R}_1 - \mathcal{R}_1\rho\}_{12}, a_3b_2) = -\tfrac{1}{4}g_{34/22}^2,$$

$$c(\{\rho\mathcal{J}_1 - \mathcal{J}_1\rho\}_{21}, v_1v_4) = \tfrac{1}{4}g_{34/11}^2,$$

$$c(\{\rho\mathcal{J}_1 - \mathcal{J}_1\rho\}_{12}, v_2v_3) = \tfrac{1}{4}g_{34/22}^2,$$

$$c(\{\mathcal{R}_1\mathcal{R}_2 - \mathcal{R}_2\mathcal{R}_1\}_{21}, a_4b_1c_3d_1) = -\tfrac{1}{4}g_{34/11}^2,$$

$$c(\{\mathcal{R}_1\mathcal{R}_2 - \mathcal{R}_2\mathcal{R}_1\}_{12}, a_4b_2c_3d_2) = \tfrac{1}{4}g_{34/22}^2.$$

Consequently, if $\mathcal{M}$ is Einstein or curvature-Ricci commuting or Jacobi–Ricci commuting or curvature–curvature commuting, we have $g_{34/11} = g_{34/22} = 0$ so

$$g_{34} = x_1 p(x_3, x_4) + x_2 q(x_3, x_4) + x_1 x_2 r(x_3, x_4) + s(x_3, x_4).$$

We then compute

$$c\left(\rho_{13}, x_1^2\right) = -\tfrac{1}{2}r(x_3, x_4)^2,$$

$$c(\{\rho\mathcal{R}_1 - \mathcal{R}_1\rho\}_{13}, a_4b_3x_1^3x_2) = -\tfrac{1}{4}r(x_3, x_4)^4,$$

$$c\big(\{\rho\mathcal{J}_1 - \mathcal{J}_1\rho\}_{13}, v_3 v_4 x_1^3 x_2\big) = \tfrac{1}{2} r(x_3, x_4)^4,$$
$$c\big(\{\mathcal{R}_1\mathcal{R}_2 - \mathcal{R}_2\mathcal{R}_1\}_{13}, a_1 b_4 c_4 d_3 x_1^2 x_2^2\big) = -\tfrac{1}{4} r(x_3, x_4)^4.$$

Consequently, if $\mathcal{M}$ is Einstein or curvature-Ricci commuting or Jacobi–Ricci commuting or curvature–curvature commuting, we have $g_{34/12} = 0$ so g_{34} is affine in $\{x_1, x_2\}$ and has the form

$$g_{34} = x_1 p(x_3, x_4) + x_2 q(x_3, x_4) + s(x_3, x_4). \tag{2b}$$

3. The Osserman condition—the proof of theorem 1.2

Assertions (1) and (2) of theorem 1.2 follow from work of [5]. It is immediate that (3a) implies (3b). Suppose (3b) holds so $\mathcal{M}$ is Einstein and g_{34} has the form of equation (2b). One computes that

$$\rho = \begin{pmatrix} 0 & 0 & -\tfrac{1}{2}q^2 + q_{/3} & \tfrac{1}{2}(pq - q_{/4} - p_{/3}) \\ 0 & 0 & \tfrac{1}{2}(pq - q_{/4} - p_{/3}) & -\tfrac{1}{2}p^2 + p_{/4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The equivalence of assertions (3b), (3c) and (3d) in theorem 1.2 now follows. Suppose any of these holds. By assertion (1), $\mathcal{M}$ is conformally Osserman. Since $\rho = 0$, $\mathcal{M}$ is Osserman.

4. The proof of theorem 1.4

A direct computation shows that if g_{34} has the form given in (3) of theorem 1.4, then $\mathcal{M}$ is both curvature-Ricci commuting and Jacobi–Ricci commuting. Furthermore, $\mathcal{R}\rho$ and $\mathcal{J}\rho$ are generically nonzero. We adopt the notation of equations (2a) and (2b).

$$c\big(\{\rho\mathcal{J}_1 - \mathcal{J}_1\rho\}_{14}, v_4^2\big) = \tfrac{1}{8}(p^2 - 2p_{/4})(q_{/4} - p_{/3}),$$
$$c\big(\{\rho\mathcal{J}_1 - \mathcal{J}_1\rho\}_{14}, v_3^2\big) = \tfrac{1}{8}(q^2 - 2q_{/3})(q_{/4} - p_{/3}),$$
$$c(\{\rho\mathcal{J}_1 - \mathcal{J}_1\rho\}_{14}, v_3 v_4) = -\tfrac{1}{4}(pq - q_{/4} - p_{/3})(q_{/4} - p_{/3}),$$
$$c(\{\rho\mathcal{R}_1 - \mathcal{R}_1\rho\}_{13}, a_3 b_4) = -\tfrac{1}{4}(q^2 - 2q_{/3})(q_{/4} - p_{/3}),$$
$$c(\{\rho\mathcal{R}_1 - \mathcal{R}_1\rho\}_{14}, a_3 b_4) = \tfrac{1}{4}(pq - q_{/4} - p_{/3})(q_{/4} - p_{/3}),$$
$$c(\{\rho\mathcal{R}_1 - \mathcal{R}_1\rho\}_{24}, a_3 b_4) = -\tfrac{1}{4}(p^2 - 2p_{/4})(q_{/4} - p_{/3}).$$

Suppose either that $\mathcal{M}$ is Jacobi–Ricci commuting or that $\mathcal{M}$ is curvature-Ricci commuting. We assume that $q_{/4} \neq p_{/3}$ and argue for a contradiction. The relations given above then show $p^2 = 2p_{/4}, q^2 = 2q_{/3}$ and $pq = q_{/4} + p_{/3}$. We now use lemma 1.3 in a crucial fashion to see that this implies $q_{/4} = p_{/3} = \tfrac{1}{2}pq$ which is contrary to our assumption.

5. The proof of theorem 1.5

We begin by studying curvature–curvature commuting manifolds. A direct computation shows that if g_{34} has the form given in theorem 1.5 (1b), then $\mathcal{M}$ is curvature–curvature commuting.

Suppose conversely that $\mathcal{M}$ is curvature–curvature commuting. Again, we adopt the notation of equations (2a) and (2b). We compute

$$c(\{\mathcal{R}_1\mathcal{R}_2 - \mathcal{R}_2\mathcal{R}_1\}_{14}, a_3b_1c_4d_3) = \tfrac{1}{8}(pq - 2p_{/3})(q_{/4} - p_{/3}),$$
$$c(\{\mathcal{R}_1\mathcal{R}_2 - \mathcal{R}_2\mathcal{R}_1\}_{23}, a_4b_2c_3d_4) = -\tfrac{1}{8}(pq - 2q_{/4})(q_{/4} - p_{/3}).$$

This implies $p_{/3} = q_{/4}$.

Next, we study curvature-Jacobi commuting and Jacobi–Jacobi commuting manifolds. We polarize $\mathcal{J}$ to define $\mathcal{J}(x, y)z := \tfrac{1}{2}\{\mathcal{R}(z, x)y + \mathcal{R}(z, y)x\}$. If assertion (2$a$) holds in theorem 1.5, then $\mathcal{J}\mathcal{R} = \mathcal{R}\mathcal{J} = \mathcal{J}^2 = \mathcal{R}^2 = 0$ and $\mathcal{M}$ is curvature-Jacobi commuting and Jacobi–Jacobi commuting (see [8]). Suppose that $\mathcal{M}$ is curvature-Jacobi commuting, then

$$0 = \mathcal{R}(x, y)\mathcal{J}(x)x = \mathcal{J}(x)\mathcal{R}(x, y)x = -\mathcal{J}(x)^2y$$

and $\mathcal{J}(x)^2 = 0$ for all x. If $\mathcal{M}$ is Jacobi–Jacobi commuting, then

$$0 = \mathcal{J}(x, y)\mathcal{J}(x)x = \mathcal{J}(x)\mathcal{J}(x, y)x = -\tfrac{1}{2}\mathcal{J}(x)^2y$$

and again $\mathcal{J}(x)^2 = 0$ for all x. Thus, either assertion (2b) or assertion (2c) of theorem 1.5 implies assertion (2d) of theorem 1.5 holds. If $\mathcal{J}(x)^2 = 0$, then $\rho = 0$. Finally, if $\rho = 0$, we can use theorem 1.2 (3) and once again lemma 1.3 is used to see that $g_{34} = x_1 p(x_3, x_4) + x_2 q(x_3, x_4) + s(x_3, x_4)$, where $p^2 = 2p_{/4}$, $q^2 = 2q_{/3}$ and $p_{/3} = q_{/4} = \tfrac{1}{2}pq$. We compute that

$$\mathcal{R}_1 = \begin{pmatrix} 0 & 0 & 0 & * \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{J}_1 = \begin{pmatrix} 0 & 0 & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

This verifies that theorem 1.5 (2a) holds; the equivalence of (2e) and (2f) is provided by theorem 1.2.

6. Affine extensions—the proof of theorem 1.6

Theorem 1.6 will follow from theorems 1.2–1.5 and from the following result.

Lemma 6.1. *Let $\mathcal{A}$ be as in theorem 1.6. Let $p = -2\Gamma_{34}{}^3$ and $q = -2\Gamma_{34}{}^4$.*

(1) $\rho_{\mathcal{A}}^a = 0 \Leftrightarrow p_{/3} = q_{/4}$.
(2) $\rho_{\mathcal{A}}^s = 0 \Leftrightarrow \rho_{\mathcal{A}} = 0 \Leftrightarrow \rho_{\mathcal{M}} = 0$.

Proof. Let ∇ be a torsion-free connection on $\mathbb{R}^2$ with nonzero Christoffel symbols

$$\nabla_{\partial_{x_3}}\partial_{x_4} = \nabla_{\partial_{x_4}}\partial_{x_3} = -\tfrac{1}{2}p(x_3, x_4)\partial_{x_3} - \tfrac{1}{2}q(x_3, x_4)\partial_{x_4}.$$

We compute

$$\begin{aligned}
\mathcal{R}_{\mathcal{A}}(\partial_{x_3}, \partial_{x_4})\partial_{x_3} &= \nabla_{\partial_{x_3}}\left(-\tfrac{1}{2}p\partial_{x_3} - \tfrac{1}{2}q\partial_{x_4}\right) \\
&= -\tfrac{1}{2}p_{/3}\partial_{x_3} - \tfrac{1}{2}q_{/3}\partial_{x_4} - \tfrac{1}{2}q\nabla_{\partial_{x_3}}\partial_{x_4} \\
&= \left(\tfrac{1}{4}pq - \tfrac{1}{2}p_{/3}\right)\partial_{x_3} + \left(\tfrac{1}{4}q^2 - \tfrac{1}{2}q_{/3}\right)\partial_{x_4}, \\
\mathcal{R}_{\mathcal{A}}(\partial_{x_3}, \partial_{x_4})\partial_{x_4} &= \left(\tfrac{1}{2}p_{/4} - \tfrac{1}{4}p^2\right)\partial_{x_3} + \left(\tfrac{1}{2}q_{/4} - \tfrac{1}{4}pq\right)\partial_{x_4}.
\end{aligned}$$

The Ricci tensor is then given by

$$\begin{aligned}
\rho_{\mathcal{A}}(\partial_{x_3}, \partial_{x_3}) &= \tfrac{1}{2}q_{/3} - \tfrac{1}{4}q^2, & \rho_{\mathcal{A}}(\partial_{x_3}, \partial_{x_4}) &= \tfrac{1}{4}pq - \tfrac{1}{2}q_{/4}, \\
\rho_{\mathcal{A}}(\partial_{x_4}, \partial_{x_3}) &= \tfrac{1}{4}pq - \tfrac{1}{2}p_{/3}, & \rho_{\mathcal{A}}(\partial_{x_4}, \partial_{x_4}) &= \tfrac{1}{2}p_{/4} - \tfrac{1}{4}p^2.
\end{aligned}$$

It now follows that $\rho_{\mathcal{A}}$ is symmetric $\Leftrightarrow q_{/4} = p_{/3}$; $\rho_{\mathcal{A}}$ is anti-symmetric $\Leftrightarrow 2q_{/3} = q^2, 2p_{/4} = p^2$ and $pq = q_{/4} + p_{/3}$. Finally, $\rho_{\mathcal{A}} = 0 \Leftrightarrow q^2 = 2q_{/3}, p^2 = 2p_{/4}$ and $p_{/3} = q_{/4} = \frac{1}{2}pq$. Lemma 6.1 now follows from lemma 1.3. $\qquad\square$

Acknowledgments

Appendix. A technical lemma in PDE's—the proof of lemma 1.3

If $0 \neq (a_0, a_3, a_4)$, then set

$$p := -2a_4(a_0 + a_3x_3 + a_4x_4)^{-1} \qquad \text{and} \qquad q = -2a_3(a_0 + a_3x_3 + a_4x_4)^{-1}. \tag{A.1}$$

We note that if $\lambda \neq 0$, then $(\lambda a_0, \lambda a_3, \lambda a_4)$ and (a_0, a_3, a_4) determine the same functions p and q in equation (A.1). Thus, we may regard (a_0, a_3, a_4) as belonging to the real projective space $\mathbb{RP}^2 := \{\mathbb{R}^3 - \{0\}\}/\{\mathbb{R} - \{0\}\}$. If $a_4 = 0$, then $p = 0$; if $a_3 = 0$, then $q = 0$.

Clearly, condition (3) of lemma 1.3 implies condition (2) of lemma 1.3 and condition (2) of lemma 1.3 implies condition (1) of lemma 1.3. Thus, we must show that if $p^2 = 2p_{/4}$, $q^2 = 2q_{/3}$ and $pq = p_{/3} + q_{/4}$, then p and q have the form given in equation (A.1). Set

$$\mathcal{O}_p := \{(x_1, x_2, x_3, x_4) \in \mathcal{O} : p(x_3, x_4) \neq 0\},$$
$$\mathcal{O}_q := \{(x_1, x_2, x_3, x_4) \in \mathcal{O} : q(x_3, x_4) \neq 0\}.$$

First, we suppose that $\mathcal{O}_p \cap \mathcal{O}_q$ is non-empty. Let B be a closed ball in $\mathbb{R}^4$ with non-empty interior which is contained in $\mathcal{O}$ and which has $\text{int}(B) \subset \mathcal{O}_p \cap \mathcal{O}_q$. We integrate the equation $p^2 = 2p_{/4}$ on $\text{int}(B)$ to express

$$p(x_3, x_4) = -2(\xi(x_3) + x_4)^{-1} \qquad \text{on} \quad \text{int}(B). \tag{A.2}$$

We use the relation $pq = p_{/3} + q_{/4}$ to conclude

$$-2(\xi(x_3) + x_4)^{-1}q = 2\dot{\xi}(x_3)(\xi(x_3) + x_4)^{-2} + q_{/4}(x_3, x_4).$$

This relation can be written in the form $\{q(x_3, x_4)(\xi(x_3) + x_4)^2\}_{/4} = -2\dot{\xi}(x_3)$. Consequently,

$$q(x_3, x_4) = \{\phi(x_3) - 2\dot{\xi}(x_3)x_4\}(\xi(x_3) + x_4)^{-2}. \tag{A.3}$$

We set $q^2 = 2q_{/3}$ and clear denominators to obtain the relation

$$\{\phi(x_3) - 2\dot{\xi}(x_3)x_4\}^2 = 2\{\dot{\phi}(x_3) - 2\ddot{\xi}(x_3)x_4\}(\xi(x_3) + x_4)^2$$
$$- 2\{\phi(x_3) - 2\dot{\xi}(x_3)x_4\}2\dot{\xi}(x_3)(\xi(x_3) + x_4). \tag{A.4}$$

Setting the coefficient of x_4^3 equal to zero then yields $\ddot{\xi}(x_3) = 0$ so $\xi(x_3) = \alpha_0 + \alpha_1x_3$ and equation (A.4) becomes

$$\{\phi(x_3) - 2\alpha_1x_4\}^2 = 2\dot{\phi}(x_3)(\alpha_0 + \alpha_1x_3 + x_4)^2 - 4\alpha_1(\phi(x_3) - 2\alpha_1x_4)(\alpha_0 + \alpha_1x_3 + x_4). \tag{A.5}$$

Examining the coefficient of x_4^2 in equation (A.5) shows that $\dot{\phi}(x_3) = -2\alpha_1^2$ so $\phi(x_3) = \beta_0 - 2\alpha_1^2x_3$. Equation (A.5) then further simplifies to become

$$(\beta_0 - 2\alpha_1^2x_3 - 2\alpha_1x_4)^2 = -4\alpha_1^2(\alpha_0 + \alpha_1x_3 + x_4)^2$$
$$- 4\alpha_1(\beta_0 - 2\alpha_1^2x_3 - 2\alpha_1x_4)(\alpha_0 + \alpha_1x_3 + x_4). \tag{A.6}$$

This leads to the relation $\beta_0^2 = -4\alpha_1^2\alpha_0^2 - 4\beta_0\alpha_1\alpha_0$ which implies $\beta_0 = -2\alpha_1\alpha_0$. Equations (A.2) and (A.3) now yield

$$p(x_3, x_4) = -2(\alpha_0 + \alpha_1 x_3 + x_4)^{-1},$$
$$q(x_3, x_4) = -2\alpha_1(\alpha_0 + \alpha_1 x_3 + x_4)^{-1}. \tag{A.7}$$

By continuity, equations (A.7) hold on the closed ball B and in particular p and q do not vanish on B. It now follows that $\mathcal{O} = \mathcal{O}_p = \mathcal{O}_q$. Analytic continuation now shows p and q are given by equations (A.7) on all of $\mathcal{O}$ and thus assertion (3) holds.

We therefore assume $\mathcal{O}_p \cap \mathcal{O}_q$ is empty. If $\mathcal{O}_p$ and $\mathcal{O}_q$ are both empty, then $p = q = 0$ and we may take $(a_0, a_3, a_4) = (1, 0, 0)$ to obtain a representation of the form given in (3). We therefore assume $\mathcal{O}_q$ is non-empty; the case $\mathcal{O}_p$ is non-empty and is handled similarly. Let B be a closed ball in $\mathbb{R}^4$ with non-empty interior which is contained in $\mathcal{O}$ and which satisfies $\operatorname{int}(B) \subset \mathcal{O}_q$. We integrate the equation $q^2 = 2q_{/3}$ to express

$$q = -2(x_3 + \eta(x_4))^{-1} \qquad \text{on} \quad \operatorname{int}(B).$$

Since $pq = p_{/3} + q_{/4}$ and since $p = 0$ on $\operatorname{int}(B)$, we have $\dot\eta = 0$ and hence

$$q = -2(x_3 + a)^{-1} \qquad \text{on} \quad \operatorname{int}(B). \tag{A.8}$$

Again, by continuity, this representation holds on all of B and thus q is nonzero on B. Thus, $\mathcal{O} = \mathcal{O}_q$, equation (A.8) holds on all of $\mathcal{O}$ and $p = 0$ on all of $\mathcal{O}$. This again obtains a representation for p and q of the form given in assertion (3).

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